Problem 3.32

An **anti-hermitian** (or **skew-hermitian**) operator is equal to *minus* its hermitian conjugate:

$$\hat{Q}^\dagger = -\hat{Q}. \tag{3.111}$$

(a) Show that the expectation value of an anti-hermitian operator is imaginary.

(b) Show that the eigenvalues of an anti-hermitian operator are imaginary.

(c) Show that the eigenvectors of an anti-hermitian operator belonging to distinct eigenvalues are orthogonal.

(d) Show that the commutator of two hermitian operators is anti-hermitian. How about the commutator of two *anti*-hermitian operators?

(e) Show that any operator $\hat{Q}$ can be written as a sum of a hermitian operator $\hat{A}$ and an anti-hermitian operator $\hat{B}$, and give expressions for $\hat{A}$ and $\hat{B}$ in terms of $\hat{Q}$ and its adjoint $\hat{Q}^\dagger$.

Solution

Let $\hat{A}$ be a hermitian operator $\left(\hat{A}^\dagger = \hat{A}\right)$, and let $\hat{B}$ be an anti-hermitian operator $\left(\hat{B}^\dagger = -\hat{B}\right)$. Evaluate their expectation values.

$$\begin{aligned} \langle A\rangle &= \langle \Psi \,|\, \hat{A} \,|\, \Psi\rangle \\ &= \langle \Psi \,|\, \hat{A}^\dagger \,|\, \Psi\rangle \\ &= \langle \Psi \,|\, \hat{A}^\dagger \,|\, \Psi\rangle \\ &= \left(\langle \Psi|\hat{A}^\dagger\right)\cdot|\Psi\rangle \\ &= \left[\langle\Psi|\cdot\left(\hat{A}|\Psi\rangle\right)\right]^* \\ &= \left[\langle \Psi \,|\, \hat{A} \,|\, \Psi\rangle\right]^* \\ &= \langle A\rangle^* \end{aligned} \qquad \begin{aligned} \langle B\rangle &= \langle \Psi \,|\, \hat{B} \,|\, \Psi\rangle \\ &= \langle \Psi \,|\, -\hat{B}^\dagger \,|\, \Psi\rangle \\ &= -\langle \Psi \,|\, \hat{B}^\dagger \,|\, \Psi\rangle \\ &= -\left(\langle \Psi|\hat{B}^\dagger\right)\cdot|\Psi\rangle \\ &= -\left[\langle\Psi|\cdot\left(\hat{B}|\Psi\rangle\right)\right]^* \\ &= -\left[\langle \Psi \,|\, \hat{B} \,|\, \Psi\rangle\right]^* \\ &= -\langle B\rangle^* \end{aligned}$$

Therefore, the expectation value of a hermitian operator is real, and the expectation value of an anti-hermitian operator is purely imaginary. Now consider the eigenvalue problems for $\hat{A}$ and $\hat{B}$.

$$\hat{A}|f_n\rangle = a_n|f_n\rangle \qquad \hat{B}|g_n\rangle = b_n|g_n\rangle$$

Pre-multiply both sides of the left-hand equation by the bra $\langle f_n|$, and pre-multiply both sides of the right-hand equation by the bra $\langle g_n|$.

$$\langle f_n|\cdot\left(\hat{A}|f_n\rangle\right) = \langle f_n|\cdot a_n|f_n\rangle \qquad \langle g_n|\cdot\left(\hat{B}|g_n\rangle\right) = \langle g_n|\cdot b_n|g_n\rangle$$

$$\left[\left(\langle f_n|\hat{A}^\dagger\right)\cdot|f_n\rangle\right]^* = a_n\langle f_n \,|\, f_n\rangle \qquad \left[\left(\langle g_n|\hat{B}^\dagger\right)\cdot|g_n\rangle\right]^* = b_n\langle g_n \,|\, g_n\rangle$$

Use the fact that $\hat{A}$ and $\hat{B}$ are hermitian and anti-hermitian, respectively.

$$\left[\left(\langle f_n|\hat{A}\right)\cdot|f_n\rangle\right]^* = a_n\langle f_n\,|\,f_n\rangle \qquad \left[\left(\langle g_n|\,-\hat{B}\right)\cdot|g_n\rangle\right]^* = b_n\langle g_n\,|\,g_n\rangle$$

$$\left[\langle f_n|\cdot\left(\hat{A}|f_n\rangle\right)\right]^* = a_n\langle f_n\,|\,f_n\rangle \qquad \left[-\langle g_n|\cdot\left(\hat{B}|g_n\rangle\right)\right]^* = b_n\langle g_n\,|\,g_n\rangle$$

$$\left[\langle f_n|\cdot\left(a_n|f_n\rangle\right)\right]^* = a_n\langle f_n\,|\,f_n\rangle \qquad \left[-\langle g_n|\cdot\left(b_n|g_n\rangle\right)\right]^* = b_n\langle g_n\,|\,g_n\rangle$$

$$\left[a_n\langle f_n\,|\,f_n\rangle\right]^* = a_n\langle f_n\,|\,f_n\rangle \qquad \left[-b_n\langle g_n\,|\,g_n\rangle\right]^* = b_n\langle g_n\,|\,g_n\rangle$$

$$a_n^*\langle f_n\,|\,f_n\rangle^* = a_n\langle f_n\,|\,f_n\rangle \qquad -b_n^*\langle g_n\,|\,g_n\rangle^* = b_n\langle g_n\,|\,g_n\rangle$$

$$a_n^*\langle f_n\,|\,f_n\rangle = a_n\langle f_n\,|\,f_n\rangle \qquad -b_n^*\langle g_n\,|\,g_n\rangle = b_n\langle g_n\,|\,g_n\rangle$$

$$0 = (a_n - a_n^*)\langle f_n\,|\,f_n\rangle \qquad 0 = (b_n + b_n^*)\langle g_n\,|\,g_n\rangle$$

Since the eigenvectors can't be zero, the inner products are strictly positive ($\langle f_n\,|\,f_n\rangle > 0$ and $\langle g_n\,|\,g_n\rangle > 0$).

$$0 = a_n - a_n^* \qquad 0 = b_n + b_n^*$$

$$a_n^* = a_n \qquad b_n^* = -b_n$$

Therefore, the eigenvalues of a hermitian operator are real, and the eigenvalues of an anti-hermitian operator are purely imaginary. Assume that x_n and y_n are real numbers and that $\hat{X}$ and $\hat{Y}$ are hermitian and anti-hermitian operators, respectively. Consider the eigenvalue problem of another operator $\hat{Q}$.

$$\begin{aligned}\hat{Q}|h_n\rangle &= q_n|h_n\rangle \\ &= (x_n + iy_n)|h_n\rangle \\ &= x_n|h_n\rangle + iy_n|h_n\rangle \\ &= \hat{X}|h_n\rangle + \hat{Y}|h_n\rangle \\ &= (\hat{X}+\hat{Y})|h_n\rangle\end{aligned}$$

Therefore, any operator can be written as the sum of a hermitian operator and an anti-hermitian operator.

$$\hat{Q} = \hat{X} + \hat{Y} \tag{1}$$

Take the hermitian conjugate of both sides.

$$\begin{aligned}\hat{Q}^\dagger &= \left(\hat{X}+\hat{Y}\right)^\dagger \\ &= \hat{X}^\dagger + \hat{Y}^\dagger \\ &= \hat{X} - \hat{Y}\end{aligned} \tag{2}$$

Adding the respective sides of equations (1) and (2) yields

$$\hat{Q} + \hat{Q}^\dagger = 2\hat{X},$$

whereas subtracting the respective sides of equations (1) and (2) yields

$$\hat{Q} - \hat{Q}^\dagger = 2\hat{Y}.$$

Therefore, the following combinations of $\hat{Q}$ and $\hat{Q}^\dagger$ produce a hermitian operator $\hat{X}$ and an anti-hermitian operator $\hat{Y}$.

$$\hat{X} = \frac{\hat{Q} + \hat{Q}^\dagger}{2}$$

$$\hat{Y} = \frac{\hat{Q} - \hat{Q}^\dagger}{2}$$

This can be verified.

$$\hat{X}^\dagger = \frac{\hat{Q}^\dagger + \hat{Q}}{2} = \frac{\hat{Q} + \hat{Q}^\dagger}{2} = \hat{X}$$

$$\hat{Y}^\dagger = \frac{\hat{Q}^\dagger - \hat{Q}}{2} = -\frac{\hat{Q} - \hat{Q}^\dagger}{2} = -\hat{Y}$$

Reconsider the eigenvalue problems for $\hat{A}$ and $\hat{B}$.

$$\hat{A}|f_n\rangle = a_n|f_n\rangle \qquad\qquad \hat{B}|g_n\rangle = b_n|g_n\rangle$$

Pre-multiply both sides of the left-hand equation by the bra $\langle f_m|$, and pre-multiply both sides of the right-hand equation by the bra $\langle g_m|$. Here $n \neq m$.

$$\langle f_m| \cdot \left(\hat{A}|f_n\rangle\right) = \langle f_m| \cdot a_n|f_n\rangle \qquad\qquad \langle g_m| \cdot \left(\hat{B}|g_n\rangle\right) = \langle g_m| \cdot b_n|g_n\rangle$$

$$\left[\left(\langle f_n|\hat{A}^\dagger\right) \cdot |f_m\rangle\right]^* = a_n\langle f_m \mid f_n\rangle \qquad\qquad \left[\left(\langle g_n|\hat{B}^\dagger\right) \cdot |g_m\rangle\right]^* = b_n\langle g_m \mid g_n\rangle$$

Use the fact that $\hat{A}$ and $\hat{B}$ are hermitian and anti-hermitian, respectively.

$$\left[\left(\langle f_n|\hat{A}\right) \cdot |f_m\rangle\right]^* = a_n\langle f_m \mid f_n\rangle \qquad\qquad \left[\left(\langle g_n| -\hat{B}\right) \cdot |g_m\rangle\right]^* = b_n\langle g_m \mid g_n\rangle$$

$$\left[\langle f_n| \cdot \left(\hat{A}|f_m\rangle\right)\right]^* = a_n\langle f_m \mid f_n\rangle \qquad\qquad \left[-\langle g_n| \cdot \left(\hat{B}|g_m\rangle\right)\right]^* = b_n\langle g_m \mid g_n\rangle$$

$$\left[\langle f_n| \cdot \left(a_m|f_m\rangle\right)\right]^* = a_n\langle f_m \mid f_n\rangle \qquad\qquad \left[-\langle g_n| \cdot \left(b_m|g_m\rangle\right)\right]^* = b_n\langle g_m \mid g_n\rangle$$

$$\left[a_m\langle f_n \mid f_m\rangle\right]^* = a_n\langle f_m \mid f_n\rangle \qquad\qquad \left[-b_m\langle g_n \mid g_m\rangle\right]^* = b_n\langle g_m \mid g_n\rangle$$

$$a_m^*\langle f_n \mid f_m\rangle^* = a_n\langle f_m \mid f_n\rangle \qquad\qquad -b_m^*\langle g_n \mid g_m\rangle^* = b_n\langle g_m \mid g_n\rangle$$

$$a_m^*\langle f_m \mid f_n\rangle = a_n\langle f_m \mid f_n\rangle \qquad\qquad -b_m^*\langle g_m \mid g_n\rangle = b_n\langle g_m \mid g_n\rangle$$

$$0 = (a_n - a_m^*)\langle f_m \mid f_n\rangle \qquad\qquad 0 = (b_n + b_m^*)\langle g_m \mid g_n\rangle$$

Since distinct eigenvalues are generally unrelated, $a_n - a_m^* \neq 0$ and $b_n + b_m^* \neq 0$.

$$0 = \langle f_m \mid f_n \rangle \qquad\qquad 0 = \langle g_m \mid g_n \rangle$$

Therefore, the eigenvectors of a hermitian operator are orthogonal, and the eigenvectors of an anti-hermitian operator are orthogonal. Let $\hat{A}_1$ and $\hat{A}_2$ be hermitian operators. Take the hermitian conjugate of their commutator.

$$\begin{aligned}
\left[\hat{A}_1, \hat{A}_2\right]^\dagger &= \left(\hat{A}_1\hat{A}_2 - \hat{A}_2\hat{A}_1\right)^\dagger \\
&= \left(\hat{A}_1\hat{A}_2\right)^\dagger - \left(\hat{A}_2\hat{A}_1\right)^\dagger \\
&= \left(\hat{A}_2^\dagger\hat{A}_1^\dagger\right) - \left(\hat{A}_1^\dagger\hat{A}_2^\dagger\right) \\
&= \left(\hat{A}_2\hat{A}_1\right) - \left(\hat{A}_1\hat{A}_2\right) \\
&= -\left(\hat{A}_1\hat{A}_2 - \hat{A}_2\hat{A}_1\right) \\
&= -\left[\hat{A}_1, \hat{A}_2\right]
\end{aligned}$$

Therefore, the commutator of two hermitian operators is anti-hermitian. Let $\hat{B}_1$ and $\hat{B}_2$ be anti-hermitian operators. Take the hermitian conjugate of their commutator.

$$\begin{aligned}
\left[\hat{B}_1, \hat{B}_2\right]^\dagger &= \left(\hat{B}_1\hat{B}_2 - \hat{B}_2\hat{B}_1\right)^\dagger \\
&= \left(\hat{B}_1\hat{B}_2\right)^\dagger - \left(\hat{B}_2\hat{B}_1\right)^\dagger \\
&= \left(\hat{B}_2^\dagger\hat{B}_1^\dagger\right) - \left(\hat{B}_1^\dagger\hat{B}_2^\dagger\right) \\
&= \left[\left(-\hat{B}_2\right)\left(-\hat{B}_1\right) - \left(-\hat{B}_1\right)\left(-\hat{B}_2\right)\right] \\
&= \left(\hat{B}_2\hat{B}_1\right) - \left(\hat{B}_1\hat{B}_2\right) \\
&= -\left(\hat{B}_1\hat{B}_2 - \hat{B}_2\hat{B}_1\right) \\
&= -\left[\hat{B}_1, \hat{B}_2\right]
\end{aligned}$$

Therefore, the commutator of two anti-hermitian operators is also anti-hermitian.